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Do not open the exam sheet until prompted!

From Molecular to Continuum Physics II — WS 2023/24
Exam — 25.08.2022

Permitted Aids:

- Document-proof writing utensils, but no red pen.
- Two sheets of A4 paper, handwritten on both sides, with name and matriculation number.
- Other aids, especially the use of a calculator, are not allowed.

Hints:

- The use of cell phones during the exam is considered as an attempt to cheat.
- You have a total of **120 minutes** to complete the exam. *All answers must be justified in detail.*
- To pass the exam, **50%** of the possible points are sufficient.
- Please begin each assignment on the sheet on which the assignment is formulated. If you use any of the attached blank sheets in addition to the blank page opposite, please indicate “Continue on another sheet” on the first sheet. *Please mark each sheet with your name and matriculation number — even the blank sheets used.*
- By signing this form, you assure that, to the best of your knowledge, you are fit to take the exam at the beginning of the exam and that the exam performance was performed by you without any unauthorized aids.

Matriculation number: ___ ___ ___ ___ ___ ___

Last name, first name: ______________________________

Signature: ______________________________

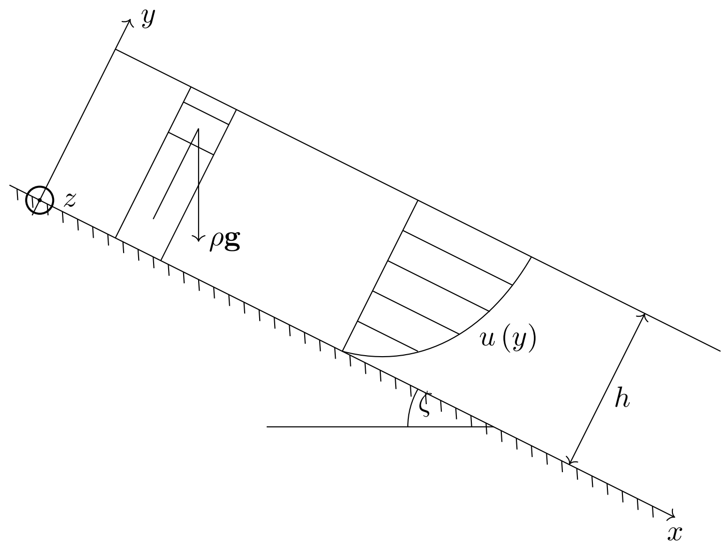
Problem	1	2	3	4	5	$\sum$
Points	20	20	20	20	20	100
Your points						

Exam [ ] + Bonus [ ] = Sum [ ]

Grade: [ ]

Problem 1.

Consider a shallow film flow on an inclined plane. We do not specify a constitutive law yet.



a) Write down the balance laws,

$$\frac{1}{\rho}\frac{D}{Dt}\rho + \nabla \cdot \mathbf{v} = 0$$
$$\rho \frac{D}{Dt}\mathbf{v} = -\nabla p + \nabla \cdot \tau + \rho \mathbf{g},$$

in components while writing out the material derivative and assuming that the coordinate system is tilted with respect to the horizontal by an angle of ζ. Use this fact to explicitly write out the term $\rho\mathbf{g}$.

b) Assuming i) constant density, ii) steady flow, iii) no change in the velocity profile along the inclined plane, and iv) a quasi 2-D situation (no velocity gradient in cross flow direction, z) how do the equations look like?

c) Determine the pressure profile in y-direction and make a sketch, assuming pressure-free conditions at the free surface. How does the pressure profile change as we increase the inclination angle?

d) Determine the velocity profile in y-direction and make a sketch, assuming no-slip conditions at the base and stress-free conditions at the free surface (vanishing strain). How does the velocity profile change as we increase the inclination angle?

e) If we derive a depth-averaged model from these equations, we can use a shape factor to account for the velocity profile. Write down how the shape factor is defined and compute it for the derived velocity profile as a function of the inclination angle.

20 points

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Problem 2.

Let's consider a shallow flow with a non-constant velocity profile that can be compensated for by shape factor $\alpha \geq 1$, such that the process model reads

$$\frac{\partial}{\partial t}h + \frac{\partial}{\partial x}(hu) = 0$$
$$\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\left(\alpha h u^2 + g\frac{h^2}{2}\right) = 0$$

a) Write the system of partial differential equations in vector notation using $\begin{pmatrix} h \\ hu \end{pmatrix}$.

b) Compute the characteristic speeds by
   (I) Substituting for $q := hu$,
   (II) computing the flux Jacobian and substitute back,
   (III) determining the characteristic polynomial, and
   (IV) finding its roots.

c) Classical shallow flow is supercritical, if both characteristic speeds have the same sign. For simplicity, you can think of both being positive. For a "plug flow regime", in which the shape factor equals to 1, *supercritical* corresponds to the condition $\mathrm{Fr} = \frac{u}{\sqrt{gh}} > 1$. Now, we are interested how the condition changes for a shape factor that differs from one.

   Assume that the shape factor is always larger than one. Explain what happens as the shape factor increases. How does the condition read for a linear velocity profile, i.e. $\alpha = \frac{4}{3}$?

d) What is the condition for supercritical flow as a function of the general shape factor? Interpret how the condition changes with α.

20 points

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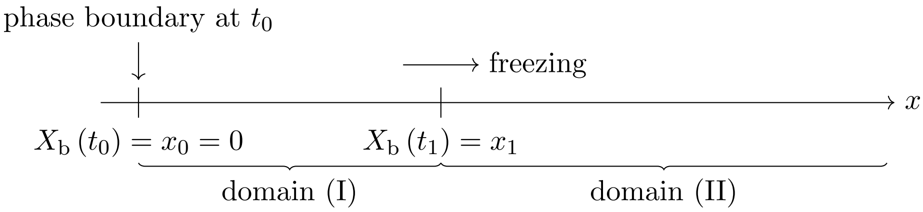
Problem 3.

Let's assume, you have a box with cold water at the melting temperature $T_\mathrm{m} = 0\,°\mathrm{C}$. You cool down the left side ($x_0 = 0$) of the box at a constant temperature $T_\mathrm{wall} = -10\,°\mathrm{C}$ and you are interested in how fast ice grows from the left side with time. If we assume that the density of ice and water is the same, we can model this in 1-D:



a) What is the difference between domain (I) and domain (II)?

b) State the transient heat equation including boundary conditions for both domains. Is the problem already closed? If not, what is the missing condition (both name and equation)? Write down and interpret physically.

c) Show that for fixed interface position X_b the following temperature profile

$$T(x,t) = T_\mathrm{wall} - (T_\mathrm{wall} - T_\mathrm{m}) \frac{\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)}{\mathrm{erf}\left(\frac{X_\mathrm{b}}{2\sqrt{\alpha t}}\right)}, \tag{1}$$

with the error function defined as

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp\left(-\tilde{z}^2\right) \mathrm{d}\tilde{z},$$

solves the process model in domain (I) as well as the boundary conditions from task b) (not the interface condition).

d) Assume the ansatz

$$X_\mathrm{b}(t) = 2\sqrt{\alpha}\lambda\sqrt{t}.$$

Substitute the ansatz and (1) into the interface condition from task b). Simplify the result by using

$$\mathrm{Ste} = \frac{L}{c_p\left(T_\mathrm{wall} - T_\mathrm{m}\right)},$$

where L is the latent heat of melting, and c_p is the specific heat capacity. Specify the units and interpret Ste physically. Write as an homogeneous function ($F(\lambda) = 0$). Sketch F. How many roots does F have?

e) *Qualitatively* sketch the position of the interface as a function of time and the temperature profile at two (arbitrary) time instants (no quantitative assessment necessary!).

f) What happens with the temperature profile as T_wall gets colder (e.g., $-20\,°\mathrm{C}$)? What happens with the interface position at a specific time in comparison to the previous case?

20 points

Problem 4.

The 1-D elastic wave equation for shear waves (S-waves) in *velocity-stress formulation* is given by

$$\frac{\partial}{\partial t}\mathbf{Q} - \mathbf{A}\frac{\partial}{\partial x}\mathbf{Q} = \mathbf{0}, \tag{2}$$

where

$$\mathbf{Q} = \begin{pmatrix} \sigma \\ v \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & \mu \\ \rho^{-1} & 0 \end{pmatrix},$$

with σ denoting the stress and v denoting the particle velocity. μ is the shear modulus and ρ is the density. Consider a harmonic plane wave propagating in x-direction given by

$$\mathbf{Q}\,(x,t) = \mathbf{Q}_0 \exp\left[\mathrm{i}\,(kx - \omega t - \varphi)\right], \tag{3}$$

with $\mathbf{Q}_0 = \begin{pmatrix} \sigma_0 \\ v_0 \end{pmatrix} = \text{const}.$

a) In which direction(s) does the oscillation take place? What is actually oscillating here?

b) Sketch the real part of the velocity component of $\mathbf{Q}\,(x_0, t)$ over time for a fixed position x_0. Name the axes in the sketch. Use the sketch to explain amplitude, period, frequency and phase of the wave.

c) Insert the ansatz (3) into the wave equation (2) and calculate the wave speeds $c = \frac{\omega}{k}$. Explain the physical meaning of your result.

d) Assume you want to model wave propagation in porous media. For example, consider ultrasonic measurements on samples of porous sandstone that you have collected. Now, if you want to save computational time and model this using the 3-D elastic wave equation rather than the full 3-D poroelastic wave equation, what physical phenomena will you actually neglect? Explain.

20 points

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Problem 5.

You have given the incompressible Navier-Stokes-Fourier system in Boussinesq formulation:

$$\begin{aligned}\nabla \cdot \mathbf{v} &= 0 \\ \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\frac{1}{\rho_0} \nabla (p - \rho_0 g z) + \nu \triangle \mathbf{v} - \mathbf{g} B (T - T_0) \\ \partial_t (\rho_0 c_p T) + \nabla \cdot (\rho_0 c_p T \mathbf{v}) &= \nabla \cdot (\kappa \nabla T) + \mathbf{S}.\end{aligned}$$

a) Name all unknowns and parameters including their units.

b) Introduce the following scales,
   - Length scale L,
   - Velocity scale V,
   - Time scale $\frac{L}{V}$, and
   - Pressure scale $\mu \frac{V}{L}$,

   and re-write the momentum equation in a dimensionless formulation, such that the viscosity term is scaled to one. Stay with the vector notation for conveniency.

   Which dimensionless numbers remain in the system?

c) Derive the dimensionless temperature equation. Note, that you can make use of the Peclet number Pe, which is defined as $\mathrm{Pe} = L\frac{V}{\alpha}$, with α being the thermal diffusivity.

d) The Prandtl number (Pr) denotes the ratio between momentum diffusivity ν and the thermal diffusivity α. Write the dimensionless temperature equation in terms of the Prandtl number.

   Hint: First derive an expression of the Pr number in terms of Re and Pe

e) Determine the values for the dimensionless quantities Re, Pe, and Pr for lubricated flow in a very small gap ($L \approx 10\,\mathrm{mm}$) and for expected velocities below $10^{-3}\,\mathrm{m\,s^{-1}}$. The momentum diffusivity of oil is $\nu \approx 10\,\mathrm{mm^2\,s^{-1}}$ and the thermal diffusivity of oil is $\alpha = 0.0738\,\mathrm{mm^2\,s^{-1}}$.

   Based on the values of the dimensionless numbers in this situation, discuss which components of the Navier-Stokes equations can be neglected.

20 points

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